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Representations of orthosymplectic superalgebras: II. Young diagrams and weight space techniques

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Abstract. Finite-dimensional graded tensor representations of $\mathrm{OSp}(M/N)$ are enumerated via standard Young diagrams, and their corresponding highest weight and Kac–Dynkin labels are given. A uniform set of conditions on the diagram shape, necessary and sufficient for atypicality, are presented. Weight space techniques are used to provide a complete analysis of all atypical representations for the low-rank cases OSp(2/2), OSp(3/2) and OSp(4/2).

1. Introduction and main results

Continued recent interest in supersymmetries in physics has underlined the importance of developing the supporting mathematical framework, in particular for the classical superalgebras. This paper continues a study of representations of orthosymplectic superalgebras begun in Farmer and Jarvis (1983, hereafter referred to as I) wherein are given references to a variety of physical applications (for most recent developments see, for example, Bars 1982a, Efetov 1983, and Duff 1983).

The methods of I using superfield (functions on supermanifolds) realisations of representations are complemented in the present work by attention to weight space techniques and to the graded Young diagrams for orthosymplectic superalgebras. Since the original work (Dondi and Jarvis 1980, 1981, Balantekin and Bars 1981), several reviews have appeared on supertableaux for the unitary cases $\mathrm{SU}(m+1/n+1)\simeq A(m, n)$ (Bars 1982b, Bars *et al* 1982; see also Delduc and Gourdin 1982) and similar ground is covered here (see also King 1982) for the orthosymplectic case $\mathrm{OSp}(M/N)$ which includes the classes $\mathrm{OSp}(1/2n)\equiv B(o, n)$, $\mathrm{OSp}(2/2n-2)\equiv C(n)$, $\mathrm{OSp}(2m+1/2n)\equiv B(m, n)$ and $\mathrm{OSp}(2m/2n)\equiv D(m, n)$ (for the classification of superalgebras see Kac 1977, Rittenberg 1978, Scheunert 1979).

As in I the focus of the work is to provide concrete constructions in specific cases of use for physical applications and also within which general open questions regarding, for example, atypical and indecomposable representations may be answered explicitly. Thus in § 3 standard Young diagrams for $\mathrm{OSp}(M/N)$ are enumerated and the corresponding Kac–Dynkin labels given. This allows the atypicality conditions of Kac (1978) to be formulated as constraints on the diagram shape (table 2). In § 4 an explicit weight-space construction is used to provide a complete analysis of all finite-dimensional irreducible representations of the lowest rank members of each of the

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classes, namely $C(2)$, $B(1,1)$ and $D(2,1)$. In particular the structure of the atypical representations is elucidated (table 3). The formalism also entails the consideration of inner products and (grade) star conditions. The results are negative in that finite-dimensional irreducible grade star representations on a graded Hilbert space (i.e. with positive definite inner product) exist for only a few cases in the examples studied (§ 4.2).

In § 2 we establish the notation and review the structure of orthosymplectic superalgebras with tabulations of the Cartan matrices, root systems and generators in a weight basis (Kac 1977, 1978, Hurni and Morel 1982, cf. Bars *et al* 1982, Hurni and Morel 1983, Tits 1967).

2. Structure of superalgebras

In this section we present the algebras for $B(o, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$ which we will be using in the succeeding sections. The root systems, Dynkin diagrams and Cartan matrices have been given by Kac (1978). For completeness we present them here. $C(n)$ belongs to the class I superalgebras which can be decomposed as $G = G_{-1} \oplus G_0 \oplus G_1$ with $[G_i, G_j] \subset G_{i+j}$. $B(o, n)$, $B(m, n)$ and $D(m, n)$ belong to the class II superalgebras which can be decomposed as $G = G_0 \oplus G_1$ with $[G_1, G_1] \subset G_0$. The even (odd) roots we designate as $\Delta_0(\Delta_1)$. For $\mathrm{OSp}(M/N)$ the roots are expressed in terms of linear functions $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{[M/2]}, \delta_1, \delta_2, \ldots, \delta_{N/2}$ which form a unit basis of H^*, the dual space of the Cartan subalgebra H, with inner product $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta_k, \delta_l) = -\delta_{kl}$, $(\varepsilon_i, \delta_k) = 0$, where $1 \leqslant i, j \leqslant [M/2]$, $1 \leqslant k, l \leqslant N/2$ ($[M/2]$ is the integer part of $M/2$).

We work in the basis provided by Kac (1978) which can be written in the following form: if h_i ($i = 1, 2, \ldots, r$; $r =$ rank of the superalgebra) are the generators of the Cartan subalgebra and $\alpha_i^+(\alpha_i^-)$ are the generators corresponding to the ith positive (negative) simple root, then

$$[\alpha_i^+, \alpha_j^-] = \delta_{ij} h_i \qquad [h_i, h_j] = 0 \qquad [h_i, \alpha_j^\pm] = \pm a_{ij} \alpha_j^\pm$$

where the a_{ij} are elements of the Cartan matrix. The remaining generators may be defined from the simple roots by (anti-)commutation (cf. Tits 1967, Hurni and Morel 1982, 1983).

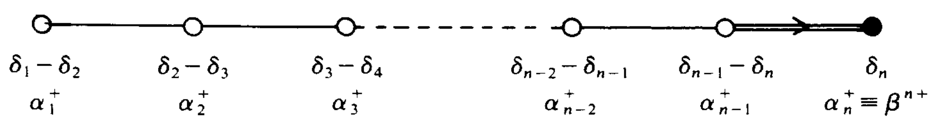
The weight space decomposition of a representation is given by the eigenvalues a_i' and b' of a vector with respect to h_i and k respectively. The so-called 'hidden' Cartan generator, k, is defined by equations (2.1), (2.2), (2.3) and (2.4) for $B(o, n)$, $B(m, n)$, $D(m, n)$ and $C(n)$ respectively. We label the highest weight vector of a representation by a_i and b.

2.1. $B(o, n)$

$$\Delta_0 = \{\pm 2\delta_k, \pm\delta_k \pm \delta_l\} \qquad k \neq l$$

$$\Delta_1 = \{\pm\delta_k\} \qquad 1 \leqslant k, l \leqslant n.$$

The Dynkin diagram with the set of simple positive roots chosen and their associated generators is



The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix}.$$

The remaining odd generators we construct in the following way

$$\beta^{i\pm} = [[\dots[[\beta^{n\pm}, \alpha_{n-1}^{\pm}], \alpha_{n-2}^{\pm}], \dots], \alpha_i^{\pm}]$$

where $1 \leqslant i \leqslant n-1$. The Sp($2n$) generator, k, in the Cartan subalgebra we take as

$$k = \tfrac{1}{2} h_n. \tag{2.1}$$

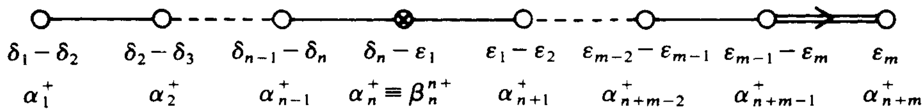
The 'hidden' Sp($2n$) generator associated with the nth node of the Dynkin diagram we take as $\{\beta^{n\pm}, \beta^{n\pm}\}$.

2.2. B(m, n) m > 0

$$\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm 2\delta_k, \pm\delta_k \pm \delta_l\} \qquad i \neq j, k \neq l$$

$$\Delta_1 = \{\pm\delta_k, \pm\varepsilon_i \pm \delta_k\} \qquad 1 \leqslant i, j \leqslant m; 1 \leqslant k, l \leqslant n.$$

The Dynkin diagram is



where we have given the set of positive simple roots chosen and the generator associated with each. The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix} 2 & -1 & & & & & & & & & & & \\ -1 & 2 & -1 & & & & & & & & & & \\ & -1 & 2 & -1 & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & -1 & 2 & -1 & & & & & & \\ & & & & & -1 & 0 & +1 & & & & & \\ & & & & & & -1 & 2 & -1 & & & & \\ & & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & -1 & 2 & -1 \\ & & & & & & & & & & & -2 & 2 \end{bmatrix}.$$

We construct the remaining odd generators in the following way

$$\beta_n^{i\pm} = [[\dots[[\beta_n^{n\pm}, \alpha_{n-1}^{\pm}], \alpha_{n-2}^{\pm}], \dots], \alpha_i^{\pm}]$$

$$\beta_j^{i\pm} = [[\dots[[\beta_n^{i\pm}, \alpha_{n+1}^{\pm}], \alpha_{n+2}^{\pm}], \dots], \alpha_j^{\pm}]$$

$$\tilde{\beta}_j^{i\pm} = [[\dots[[\beta_{n+m}^{i\pm}, \alpha_{n+m}^{\pm}], \alpha_{n+m-1}^{\pm}], \dots], \alpha_j^{\pm}]$$

where $1 \leqslant i \leqslant n$; $n+1 \leqslant j \leqslant n+m$.

There exists also a 'hidden' generator, k, of the Cartan subalgebra of $\mathrm{Sp}(2n)$. k will be some linear combination of the h_i's which satisfies the requirements

$$[k, \alpha_j^{\pm}] = 0 \qquad n+1 \leqslant j \leqslant n+m$$
$$[k, \alpha_{n-1}^{\pm}] = \mp \alpha_{n-1}^{\pm}$$
$$[k, \{\beta, \beta\}^{\pm}] = \pm 2\{\beta, \beta\}^{\pm}$$

where $\{\beta, \beta\}$ refers to one of the 'hidden' generators given below. We find

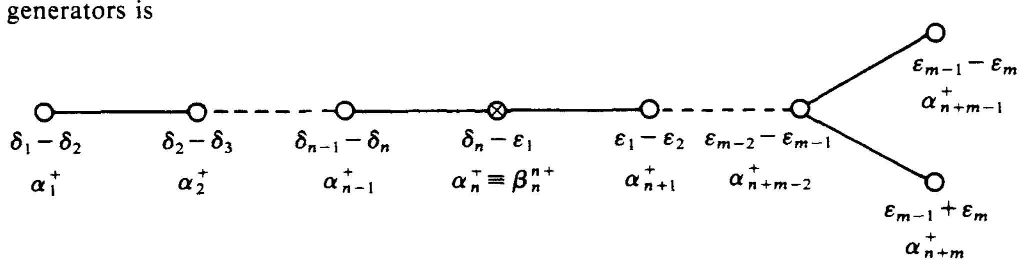
$$k = h_n - h_{n+1} - h_{n+2} - \ldots - h_{n+m-1} - \tfrac{1}{2} h_{n+m}. \tag{2.2}$$

Associated with the nth node of the Dynkin diagram we also have a 'hidden' $\mathrm{Sp}(2n)$ generator which in the basis chosen can be taken as one of $\{\beta_j^{n\pm}, \tilde{\beta}_{j+1}^{n\pm}\}$ where $n \leqslant j \leqslant n+m-1$ or as $\{\beta_{n+m}^{n\pm}, \beta_{n+m}^{n\pm}\}$.

2.3. $D(m, n)$

$$\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j\,;\ \pm 2\delta_k\,;\ \pm\delta_k \pm \delta_l\} \qquad i \neq j,\ k \neq l$$
$$\Delta_1 = \{\pm\varepsilon_i \pm \delta_k\} \qquad 1 \leqslant i, j \leqslant m;\quad 1 \leqslant k, l \leqslant n.$$

The Dynkin diagram with the set of simple positive roots chosen and their associated generators is



The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix}
2 & -1 & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & -1 & 2 & -1 & & & & & & \\
 & & & & -1 & 0 & +1 & & & & & \\
 & & & & & -1 & 2 & -1 & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & -1 & 2 & -1 & & \\
 & & & & & & & & -1 & 2 & -1 & -1 \\
 & & & & & & & & & -1 & 2 & 0 \\
 & & & & & & & & & -1 & 0 & 2
\end{bmatrix}.$$

We construct the remaining odd generators in the following way:

$$\beta_n^{i\pm} = [[\ldots[[\beta_n^{n\pm}, \alpha_{n-1}^{\pm}], \alpha_{n-2}^{\pm}], \ldots], \alpha_i^{\pm}]$$
$$\beta_j^{i\pm} = [[\ldots[[\beta_n^{i\pm}, \alpha_{n+1}^{\pm}], \alpha_{n+2}^{\pm}], \ldots], \alpha_j^{\pm}]$$
$$\beta_{n+m}^{i\pm} = [\beta_{n+m-2}^{j\pm}, \alpha_{n+m}^{\pm}]$$

$$\tilde{\beta}_j^{i\pm} = [[\dots[[\beta^{i\pm}_{n+m}, \alpha^{\pm}_{n+m-1}], \alpha^{\pm}_{n+m-2}], \dots], \alpha^{\pm}_j]$$

$$n+1 \leqslant j \leqslant n+m-1; \quad 1 \leqslant i \leqslant n.$$

The 'hidden' Sp($2n$) generator in the Cartan subalgebra is

$$k = h_n - h_{n+1} - h_{n+2} - \dots - h_{n+m-2} - \tfrac{1}{2}(h_{n+m-1} + h_{n+m}). \tag{2.3}$$

The 'hidden' Sp($2n$) generator associated with the nth Dynkin node can be taken as one of

$$\{\beta_j^{n\pm}, \tilde{\beta}_{j+1}^{n\pm}\} \quad \text{where } n \leqslant j \leqslant n+m-2$$

or as

$$\{\beta^{n\pm}_{n+m-1}, \beta^{n\pm}_{n+m}\}$$

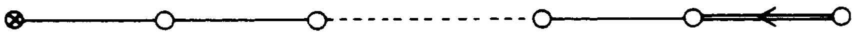
in the basis of simple roots we have chosen.

2.4. $C(n)$

$$\Delta_0 = \{\pm 2\delta_i, \pm\varepsilon_1 \pm \delta_j\} \qquad i \neq j$$

$$\Delta_1 = \{\pm\varepsilon_1 \pm \delta_i\} \qquad 1 \leqslant i, j \leqslant n-1.$$

The Dynkin diagram, with the simple positive roots and their associated generators shown, is†



$$\begin{array}{cccccc} \varepsilon_1 - \delta_1 & \delta_1 - \delta_2 & \delta_2 - \delta_3 & \delta_{n-3} - \delta_{n-2} & \delta_{n-2} - \delta_{n-1} & 2\delta_{n-1}\,. \\ \alpha_1^+ \equiv \beta_1^+ & \alpha_2^+ & \alpha_3^+ & \alpha_{n-2}^+ & \alpha_{n-1}^+ & \alpha_n^+ \end{array}$$

The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix} 0 & +1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & & & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -2 \\ & & & & & & -1 & 2 \end{bmatrix}.$$

We construct the remaining odd generators in the following way

$$\beta_i^{\pm} = [[\dots[[\beta_1^{\pm}, \alpha_2^{\pm}], \alpha_3^{\pm}], \dots], \alpha_i^{\pm}] \qquad 2 \leqslant i \leqslant n$$

$$\tilde{\beta}_j^{\pm} = [[\dots[[\beta_n^{\pm}, \alpha_{n-1}^{\pm}], \alpha_{n-2}^{\pm}], \dots], \alpha_j^{\pm}] \qquad 2 \leqslant j \leqslant n-1$$

The 'hidden' O(2) generator is

$$k = h_1 - h_2 - h_3 - \dots - h_n. \tag{2.4}$$

3. Young supertableaux and atypicality conditions

In this section we examine finite-dimensional representations of OSp(M/N) via standard Young diagrams. These can be realised by graded symmetrised, supertraceless tensors (Dondi and Jarvis 1981). These can be decomposed in terms of irreducible representations of O(M)×Sp(N), with branching rule (King 1982),

$$[\lambda] \downarrow \sum_{\xi} [\lambda/\xi]\langle\tilde{\xi}/B\rangle = \sum_{\xi}\sum_{\beta} [\lambda/\xi]\langle\tilde{\xi}/\beta\rangle \tag{3.1a}$$

† For the $C(2) \simeq A(1, 0)$ case, see § 4.2.

or

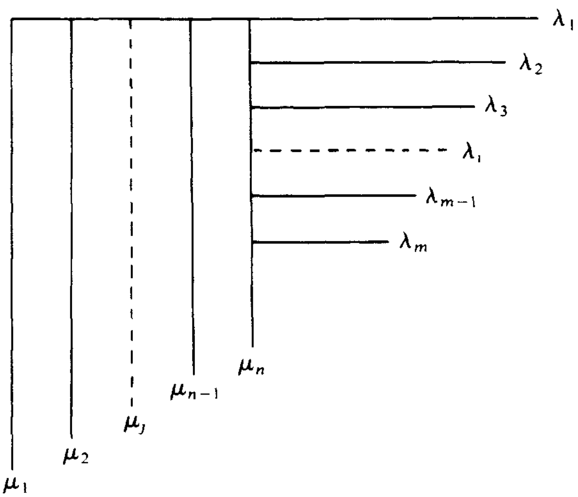
$$[\lambda]\downarrow\sum_{\xi}[\xi/D]\langle\widetilde{\lambda/\xi}\rangle=\sum_{\xi}\sum_{\delta}[\xi/\delta]\langle\widetilde{\lambda/\xi}\rangle. \tag{3.1b}$$

Here ξ runs over all divisors of λ; β corresponds to partitions with even column lengths, and δ to partitions with even row lengths.

In order to present necessary and sufficient conditions on the diagram shape for atypicality†, we examine each of $B(m, n)$, $C(n)$ and $D(m, n)$ in turn to establish the correspondence between the Kac–Dynkin labels corresponding to the highest weight and the diagram labels. The conditions for atypicality are general for $\mathrm{OSp}(M/N)$. (Further details of the weight-space realisations are given in § 4.) In the sequel we consider only 'standard' supertableaux in the following sense: for $B(m, n)$ and $D(m, n)$ the diagrams are such that if C_i is the length of the ith column then $C_i \leqslant m$ for $i > n$. for $C(n) \simeq \mathrm{OSp}(2/2n-2)$ we require $C_i \leqslant 1$ for $i > n-1$. Of course, all diagrams must be proper.

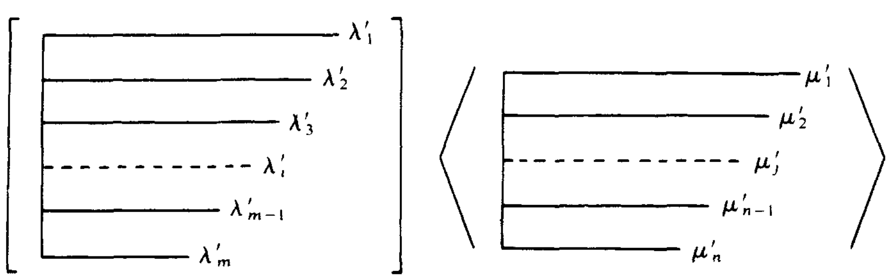
3.1. $B(m, n)$ $m \geqslant 0$

Consider the supertableau

$[\lambda]=$ 

(3.2)

where λ_i is the number of boxes beyond the nth column in the ith row, with $i \leqslant m$, and μ_j is the number of boxes in the jth column, with $j \leqslant n$. We will designate this diagram as $[\lambda_1, \lambda_2, \ldots, \lambda_m; \mu_1, \mu_2, \ldots, \mu_n]$. A general diagram in the decomposition (3.1a), after the appropriate modification, will have the form



(3.3)

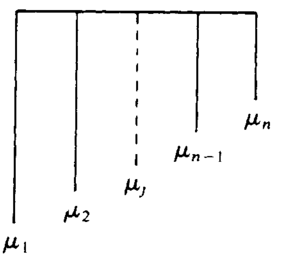
† A representation is atypical if $(\Lambda+\rho, \alpha)=0$ for some $\alpha \in \bar{\Delta}_1^+$. Here Λ is the highest weight, $\rho=\rho_0-\rho_1$ is the graded half-sum of the positive even and odd roots, and $\bar{\Delta}_1^+=\Delta_1^+\backslash\frac{1}{2}\Delta^+\cap\Delta_1^+$ (Kac 1978).

The relationships, for (3.3), between the $O(2m+1)\times Sp(2n)$ Dynkin labels and the diagram labels are given by (Black and Wybourne 1983)

$$\begin{aligned} &a'_1=\mu'_1-\mu'_2, \quad a'_2=\mu'_2-\mu'_3, \quad \ldots, \quad a'_{n-1}=\mu'_{n-1}-\mu'_n, \quad b'=\mu'_n \\ &a'_{n+1}=\lambda'_1-\lambda'_2, \quad a'_{n+2}=\lambda'_2-\lambda'_3, \quad \ldots, \quad a'_{n+m-1}=\lambda'_{m-1}-\lambda'_m, \quad a'_{n+m}=2\lambda'_m \end{aligned} \tag{3.4}$$

where $a'_1, a'_2, \ldots, a'_{n-1}$ and b' refer to the $Sp(2n)$ labels and $a'_{n+1}, a'_{n+2}, \ldots, a'_{n+m}$ refer to the $O(2m+1)$ labels.

To determine which diagram in (3.1a) corresponds to the highest weight state, Λ, with weight components $\lambda(h_i)=a_i$, $\lambda(k)=b$, of a $B(m,n)$ representation we first consider the action of the odd negative generators on Λ. The weights of these are presented in tables 1a and 1b. The action of all the odd negative generators can be obtained by considering each of those in table 1a by themselves and in conjunction with each of the even supplementary operators in table 1b. Examination of these reveals that Λ can be uniquely determined by application of the following sequence of selection criteria: (i) select those states of maximum b', (ii) within this subset select those states of maximum a'_{n-1}, (iii) select those states of maximum a'_{n-2}, etc., until we finally select the state of maximum a'_1. This state will be Λ. Expressed in diagram notation these criteria are: (i) select those diagrams of maximum μ'_n, (ii) of these select those diagrams of maximum μ'_{n-1}, etc., until we finally select the diagram with maximum μ'_1. The diagram which corresponds to Λ is obtained by taking $\beta=\{0\}$ and

$\xi=\tilde{\mu}=$ 

(3.5)

Table 1a. Weight components for some odd negative generators of $B(m,n)$ and $D(m,n)$.

	β_n^{n-}	β_n^{n-1-}	β_n^{n-2-}	β_n^{n-3-}	$\ldots$	β_n^{3-}	β_n^{2-}	β_n^{1-}
h_1						0	+1	−1
h_2						+1	−1	0
h_3						−1	0	
h_{n-4}				+1				
h_{n-3}			+1	−1				
h_{n-2}		+1	−1	0				
h_{n-1}	+1	−1	0	0				
h_n	0	+1	+1	+1		+1	+1	+1
(h_n)	(−2)	(0)	(0)	(0)		(0)	(0)	(0)
h_{n+1}	+1	+1	+1	+1		+1	+1	+1
h_{n+2}								
h_{n+m}								
k	−1	0	0	0		0	0	0

Table 1b. Weight components for even negative 'supplementary' generators of $B(m, n)$.

	e_1^-	e_2^-	e_3^-	$e_4^- \ldots$	e_{m-1}^-	e_m^-	$\tilde{e}_m^-$	$\tilde{e}_{m-1}^-$	$\tilde{e}_{m-2}^- \ldots$	$\tilde{e}_3^-$	$\tilde{e}_2^-$	$\tilde{e}_1^-$
h_1												
h_2												
h_{n-1}												
h_n	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−2
h_{n+1}	−2	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	−2
h_{n+2}	+1	−1	0	0	0	0	0	0	0	+1	−1	0
h_{n+3}		+1	−1	0						−1	0	
h_{n+4}			+1	−1						0		
h_{n+5}				+1								
h_{n+m-3}									+1			
h_{n+m-2}								+1	−1			
h_{n+m-1}					−1	0	+1	−1	0			
h_{n+m}					+2	0	−2	0	0			
k	0	0	0	0	0	0	0	0	0	0	0	0

Where $e_i^- = [\ldots[[\alpha_{n+1}^-, \alpha_{n+2}^-], \alpha_{n+3}^-], \ldots \alpha_{n+i}^-]$

$\tilde{e}_i^- = [\ldots[[e_m^-, \alpha_{n+m}^-], \alpha_{n+m-1}^-], \ldots \alpha_{n+i}^-]$

and $1 \leqslant i \leqslant m$.

Table 1c. Weight components for even negative 'supplementary' generators for $D(m, n)$.

	f_1^-	f_2^-	f_3^-	$f_4^- \ldots$	f_{m-2}^-	f_{m-1}^-	f_m^-	$\tilde{f}_{m-1}^-$	$\tilde{f}_{m-2}^- \ldots$	$\tilde{f}_3^-$	$\tilde{f}_2^-$	$\tilde{f}_1^-$
h_1												
h_2												
h_{n-1}												
h_n	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−2
h_{n+1}	−2	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	−2
h_{n+2}	+1	−1	0	0	0	0	0	0	0	+1	−1	0
h_{n+3}		+1	−1	0						−1	0	
h_{n+4}			+1	−1						0		
h_{n+5}				+1								
h_{n+m-3}					0	0	0	0	+1			
h_{n+m-2}					−1	0	0	+1	−1			
h_{n+m-1}					+1	−1	+1	−1	0			
h_{n+m}					+1	+1	−1	−1	0			
k	0	0	0	0	0	0	0	0	0	0	0	0

Where $f_i^- = [\ldots[[\alpha_{n+1}^-, \alpha_{n+2}^-], \alpha_{n+3}^-], \ldots \alpha_{n+i}^-]$

$f_m^- = [f_{m-2}^-, \alpha_{n+m}^-]$

$\tilde{f}_i^- = [\ldots[[f_m^-, \alpha_{n+m-1}^-], \alpha_{n+m-2}^-], \ldots \alpha_{n+i}^-]$

and $1 \leqslant i \leqslant m-1$.

Table 1d. Weight components for all negative generators of $C(n)$.

	β_1^-	β_2^-	β_3^-	β_4^-	β_{n-1}^-	β_n^-	$\tilde\beta_{n-1}^-$	$\tilde\beta_{n-2}^-$	$\tilde\beta_3^-$	$\tilde\beta_2^-$
h_1	0	−1	−1	−1	−1	−1	−1	−1	−1	−2
h_2	+1	−1	0	0	0	0	0	0	+1	−1
h_3	0	+1	−1	0	0	0	0	0	−1	0
h_4		0	+1	−1	0	0	0	0	0	0
h_5			0	+1	0	0	0	0	0	
h_{n-3}							0	+1		
h_{n-2}					0	0	+1	−1		
h_{n-1}					−1	+1	−1	0		
h_n					+1	−1	0	0		
k	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

The above tables show the weight components $\alpha(h_i)$ and $\alpha(k)$, where α are the roots associated with the indicated root vectors.

The terms in brackets in table 1a indicate the modifications necessary for consideration of $B(o, n)$.

It will be given by taking $\lambda_i' = \lambda_i$ and $\mu_j' = \mu_j$ in (3.3). Therefore the Kac–Dynkin labels a_k and b in terms of the supertableaux labels λ_i and μ_j are

$$\begin{aligned} &a_1 = \mu_1 - \mu_2, \quad a_2 = \mu_2 - \mu_3, \quad \ldots, \quad a_{n-1} = \mu_{n-1} - \mu_n, \\ &a_n = \mu_n + \lambda_1, \quad a_{n+1} = \lambda_1 - \lambda_2, \quad a_{n+2} = \lambda_2 - \lambda_3, \quad \ldots, \\ &a_{n+m-1} = \lambda_{m-1} - \lambda_m, \quad a_{n+m} = 2\lambda_m, \quad b = \mu_n. \end{aligned} \tag{3.6}$$

Using (3.6) we can now rewrite the conditions for atypicality (Kac 1978) in diagram notation. These results are given in table 2. We prove that our above choice (3.5) for ξ, uniquely determines Λ in appendix 1.

Table 2. Atypicality conditions for $\mathrm{OSp}(M/N)$.

(i)	$\mu_i + \lambda_j + N/2 = j + i - 1$
(ii)	$\mu_i + N/2 + j + 1 = \lambda_j + M + i$

Where $1 \leq i \leq N/2$: $1 \leq j \leq [M/2]$.

($[M/2]$ is the largest integer less than or equal to $M/2$.)

The diagram labelling is as given in (3.2).

For $C(n) \equiv \mathrm{OSp}(2/2n-2)$ the correlation between the above diagram labelling and that of § 3.3 is $\kappa_1 = \lambda_1 + n - 1$, and $\nu_i = \mu_i - 1$. If $\kappa_1 < n - 1$, then $\mu_i = 1$ for $\kappa_1 + 1 \leq i \leq n - 1$.

Note that for the $B(o, n)$ algebra, as defined in § 2.1, we have a direct correspondence with the above by setting $\lambda_i = 0\ \forall i$. There are no atypical representations $B(o, n)$ (Kac 1978).

3.2. $D(m, n)$

We again consider the supertableau (3.2) for which (3.1a) and (3.3) are still applicable. The relationships for (3.3) between the $\mathrm{O}(2m) \times \mathrm{Sp}(2n)$ Dynkin labels and the diagram labels are given by (Black and Wybourne 1983):

$$\begin{aligned} &a_1' = \mu_1' - \mu_2', \quad a_2' = \mu_2' - \mu_3', \quad \ldots, \quad a_{n-1}' = \mu_{n-1}' - \mu_n', \quad b' = \mu_n', \\ &a_{n+1}' = \lambda_1' - \lambda_2', \quad a_{n+2}' = \lambda_2' - \lambda_3', \quad \ldots, \quad a_{n+m-2}' = \lambda_{m-2}' - \lambda_{m-1}', \end{aligned}$$

$$
\begin{aligned}
&a^{+\prime}_{n+m-1}=\lambda'_{m-1}-\lambda'_{m}, \quad a^{-\prime}_{n+m-1}=\lambda'_{m-1}+\lambda'_{m}\\
&a^{+\prime}_{n+m}=\lambda'_{m-1}+\lambda'_{m}, \quad a^{-\prime}_{n+m}=\lambda'_{m-1}-\lambda'_{m}
\end{aligned}
\tag{3.7}
$$

where $a'_1, a'_2, \ldots, a'_{n-1}$ and b' refer to the $\mathrm{Sp}(2n)$ labels and $a'_{n+1}, a'_{n+2}, \ldots, a^{\pm\prime}_{n+m-1}, a^{\pm\prime}_{n+m}$ refer to the $\mathrm{O}(2m)$ labels. If $\lambda'_m \neq 0$, both signs arise for $a^{\pm\prime}_{n+m-1}$ and $a^{\pm\prime}_{n+m}$ corresponding to the reduction of the tensor $[\lambda]$ to a direct sum $[\lambda]_+ + [\lambda]_-$ of conjugate representations of $D(m)$.

To determine which diagram in (3.1a) corresponds to Λ we again consider the action of the odd negative generators on Λ. These weights are presented in tables 1a and 1c. The action of all the odd negative generators can be obtained by considering each of those in table 1a by themselves and in conjunction with each of the even supplementary operators in table 1c. Examination of these reveals that Λ can be uniquely determined by application of the same sequence of selection criteria as for $B(m, n)$. Consequently the diagram which corresponds to Λ is again obtained by taking $\beta = \{0\}$ and ξ as in (3.5). If $\lambda_m \neq 0$ the sign ambiguity corresponds to the decomposition of the graded tensor $[\lambda]$ into a sum of two conjugate representations of $D(m, n)$ with distinct Kac–Dynkin labels

$$
\begin{aligned}
&a_1=\mu_1-\mu_2, \quad a_2=\mu_2-\mu_3, \quad \ldots, \quad a_{n-1}=\mu_{n-1}-\mu_n,\\
&a_n=\mu_n+\lambda_1, \quad a_{n+1}=\lambda_1-\lambda_2, \quad a_{n+2}=\lambda_2-\lambda_3, \quad \ldots,\\
&a_{n+m-2}=\lambda_{m-2}-\lambda_{m-1}, \quad a^{+}_{n+m-1}=\lambda_{m-1}-\lambda_m,\\
&a^{+}_{n+m}=\lambda_{m-1}+\lambda_m, \quad a^{-}_{n+m-1}=\lambda_{m-1}+\lambda_m,\\
&a^{-}_{n+m}=\lambda_{m-1}-\lambda_m, \quad b=\mu_n.
\end{aligned}
\tag{3.8}
$$

This decomposition is the super-analogue of the $D(m)$ tensor reduction described above and is related to the outer automorphism of $D(m, n)$ generated by $\alpha^{+}_{n+m} \leftrightarrow \alpha^{+}_{n+m-1}$ for the simple roots. It is clear from table 1c that this corresponds to the usual automorphism of $D(m)$ on each irreducible representation of $\mathrm{O}(2m) \times \mathrm{Sp}(2n)$.

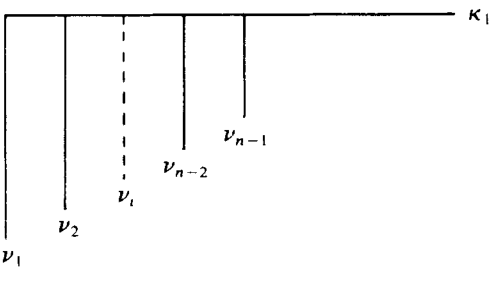
Using (3.8), the conditions for atypicality (Kac 1978) are presented in diagram notation in table 2. Note that the conditions are independent of the sign choice for $\lambda_m \neq 0$. The proof that the choice (3.5) for ξ uniquely determines Λ is given in appendix 1.

It is interesting to note that Kac's supplementary conditions, for the representation to be finite dimensional (Kac 1978) for $B(m, n)$ and $D(m, n)$, follow naturally in the above scheme.

3.3. $C(n)$

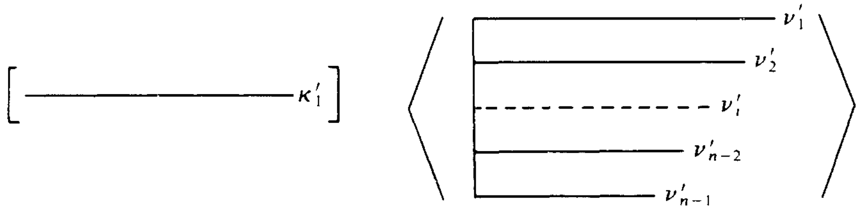
We consider here the supertableau

$[\lambda]=$



(3.9)

where κ_1 is the number of boxes in the first row and ν_j+1 is the number of boxes in the jth column, with $j \leqslant n-1$. In the decomposition (3.1b) a general diagram will take the following form, after modification,



(3.10)

The relationships, for (3.10), between the $O(2)\times Sp(2n-2)$ Dynkin labels and the diagram labels are given by (Black and Wybourne 1983):

$$\begin{aligned} &b'=\kappa_1', \quad a_2'=\nu_1'-\nu_2', \quad a_3'=\nu_2'-\nu_3', \quad \ldots, \\ &a_{n-1}'=\nu_{n-2}'-\nu_{n-1}', \quad a_n'=\nu_{n-1}' \end{aligned} \tag{3.11}$$

where b' is the O(2) label and $a_2', \ldots, a_n'$ the $Sp(2n-2)$ labels. Since the branching rule for $O(2)\downarrow U(1)\simeq SO(2)$ is $[b']\downarrow\{b'\}+\{\bar{b}'\}$ (King 1975), then an O(2) tensor $[\lambda]$ with Dynkin label b' will decompose into a direct sum $[\lambda]_+ + [\lambda]_-$ with Dynkin labels $b'^+=+b'$ and $b'^-=-b'$ respectively.

To determine which diagram in (3.1b) corresponds to Λ_1 we again consider the action of the odd negative generators on Λ. These results are presented in table 1d. Examination of these reveals that the $O(2)\times Sp(2n-2)$ highest weight state of maximum b' must be Λ. The diagram (3.10) which corresponds to Λ must, therefore, have $\kappa_1'=\kappa_1$. This state is unique and is obtained by taking $\xi=\{\kappa_1\}$ and $\delta=\{0\}$. For this situation (3.10) becomes $[\kappa_1]\langle\nu_1, \nu_2, \ldots, \nu_{n-1}\rangle$. To show that this is indeed the only diagram in (3.1b) containing $[\kappa_1]$ we need only show that if ξ contains more than one row, then $[\xi/D]$ contains only diagrams $[\kappa_1']$ with $\kappa_1'<\kappa_1$. This is achieved by consideration of the chain $O(2)\uparrow rU(2)\downarrow O(2)$ which diagrammatically can be expressed as $[\xi/D]\uparrow r\{\xi\}\downarrow[\xi/D]$ (King 1975). In U(2) we need consider only $\{\xi\}$. If it has more than two rows it will be zero and if it has two rows, i.e. if $\{\xi\}=\{\xi_1, \xi_2\}$, then it will have the same O(2) content as $\{\xi_1-\xi_2\}$. Thus when we consider the branching $\{\xi\}\downarrow[\xi/D]$ there will be no diagram consisting of just $[\kappa_1]$ if $\xi_2\neq 0$.

If $\kappa_1>n-1$ the graded tensor $[\lambda]$ decomposes into a sum of two conjugate representations of $C(n)$ with distinct Kac–Dynkin labels

$$\begin{aligned} &b^+=+\kappa_1, \quad b^-=2n-\kappa_1-2, \quad a_1^+=\kappa_1+\nu_1, \quad a_1^-=2n-\kappa_1-2+\nu_1, \\ &a_2=\nu_1-\nu_2, \quad a_3=\nu_2-\nu_3, \quad \ldots, \quad a_{n-1}=\nu_{n-2}-\nu_{n-1}, \quad a_n=\nu_{n-1}. \end{aligned}$$

Using these we present the atypicality conditions in table 2. Note that the conditions are independent of which of the two conjugate representations we choose to write them.

4. Weight space realisations

In this section we give a procedure for the explicit construction of irreducible representations of the orthosymplectic superalgebras by weight-space techniques. The general method follows Kac (1977, 1978); explicit results have been obtained by Hurni and Morel (1982) for several particular representations of various orthosymplectic superalgebras (see also Hurni and Morel 1983).

We first present the general formalism (§ 4.1), and then illustrate this by a complete analysis of the finite-dimensional irreducible representations of the lowest rank superalgebras from each orthosymplectic class, namely $\mathrm{OSp}(2/2) \simeq C(2)$, $\mathrm{OSp}(3/2) \simeq B(1, 1)$ and $\mathrm{OSp}(4/2) \simeq D(2, 1)$(§§ 4.2–4.4).

4.1. Formalism

We work with a representation possessing a highest weight λ, and a corresponding highest weight vector Λ such that $h_i\Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$ and $\alpha_i^+\Lambda = 0$ for all positive root vectors (see § 2 for notation). The representation is spanned by vectors of the form $\Pi_i(\alpha_i^-)^{k_i}\Lambda$; in fact, the distinct multiplets of the even subalgebra $\mathrm{O}(M)\times\mathrm{Sp}(N)$ are generated from the $2^{MN/2}$ states

$$\psi_j = (\beta_1^-)^{k_1}(\ldots)(\beta_{MN/2})^{k_{MN/2}},$$

where the β_i^- are odd negative root vectors, and $k_i = 0$ or 1, by the application of even generators.

Kac (1977, 1978) has given the conditions on a_i under which the representation is finite dimensional and irreducible (or typical). If these conditions are not satisfied, the representation is indecomposable, and the $\mathrm{O}(M)\times\mathrm{Sp}(N)$ structure of the irreducible composition factors (atypical representations) may be explicitly determined. (In fact, in certain cases some of the ψ_j belong to infinite-dimensional subspaces, and it is necessary to revert to the induced module construction (Kac 1977, see also Humphreys 1972) for an interpretation.)

For our construction it is useful to introduce an 'inner product' on the representation space. This depends on a choice of conjugation operation of the superalgebra. Following Scheunert *et al* (1977) this can be an adjoint (†), or a superadjoint (‡). Given that either exists, we have two different inner products $(\ ,\ )_{\mathrm{A}}$ or $(\ ,\ )_{\mathrm{S}}$ defined with respect to a fixed basis of the superalgebra by

$$(g_1^- g_2^- \ldots g_p^-\Lambda, f_1^- f_2^- \ldots f_q^-\Lambda)_{\mathrm{A}} = x$$

if

$$(g_p^-)^\dagger \ldots (g_2^-)^\dagger (g_1^-)^\dagger f_1^- f_2^- \ldots f_q^-\Lambda = x\Lambda$$

and

$$(g_1^- g_2^- \ldots g_p^-\Lambda, f_1^- f_2^- \ldots f_q^-\Lambda)_{\mathrm{S}} = (-1)^{\gamma_1+\gamma_2+\ldots+\gamma_p} y$$

if

$$(g_1^- g_2^- \ldots g_p^-)^\ddagger f_1^- f_2^- \ldots f_q^-\Lambda = y\Lambda,$$

and zero otherwise (i.e. if the vectors have different weights). Here g_i^-, f_j^- are negative root vectors of degrees γ_i and η_j respectively and $(g_ig_j)^\ddagger = (-1)^{\gamma_i\gamma_j} g_j^\ddagger g_i^\ddagger$, and adjoints and superadjoints are given in appendix 2. A characterisation of a vector v which belongs to an invariant subspace is that its length (v, v) should vanish (cf. Humphreys 1972, exercise 20.9); we apply this criterion to 'highest weight' vectors χ_j of the even subalgebra $\mathrm{O}(M)\times\mathrm{Sp}(N)$.

Given the ψ_j and the inner product, the first stage is to write down the χ_j by Schmidt orthogonalisation, namely (if there is no degeneracy)

$$\chi_j = \psi_j - \sum_{k<j} (\psi_j, \phi_k)\phi_k/(\phi_k, \phi_k) \tag{4.1}$$

where $\phi_k = E_k^- \chi_k$ has the same weight as ψ_j, E_k^- is an appropriate product of even negative root vectors, and $(\phi_k, \phi_k)/(\psi_j, \phi_k) \neq 0$ (these cases require separate analysis).

For the coefficients of ϕ_k in (4.1) we find that

$$(\psi_j, \phi_k)_{\mathrm{A}}/(\phi_k, \phi_k)_{\mathrm{A}} = (\psi_j, \phi_k)_{\mathrm{S}}/(\phi_k, \phi_k)_{\mathrm{S}}$$

and consequently the highest weight vector, χ_j, will be the same independent of whether it is defined using an adjoint or a grade adjoint operation. The second stage is to evaluate the lengths (χ_j, χ_j) and identify atypicality conditions and invariant subspaces.

The above construction shows that the whole representation can be made star or grade star. Indeed since the individual $(\chi_j, \chi_j)_{\mathrm{A}}$ and $(\chi_j, \chi_j)_{\mathrm{S}}$ differ at most by a sign, the crucial question is whether the representation is on a graded Hilbert space. In fact, we find no such finite-dimensional star representations for $B(m, n)$ and $D(m, n)$, but two classes for $C(2)$, depending on how the adjoint is defined, in agreement with Scheunert *et al* (1977). In the grade star case there exist two classes of finite-dimensional representations on a graded Hilbert space depending on how the superadjoint is defined, as discussed in appendix 2. These representations are given for the cases studied in the following sections.

In the examples we consider in the following sections we find that if in (4.1) $(\phi_l, \phi_l)/(\psi_j, \phi_l) = 0$, then for the procedure to be consistent (4.1) must be written as

$$\chi_j' = \psi_j - \sum_{\substack{k<j \\ k\neq l}} (\psi_j, \phi_k)\phi_k/(\phi_k, \phi_k). \tag{4.1a}$$

We find that although χ_j' is not strictly a highest weight of the even subalgebra, it is part of the infinite-dimensional invariant subspace and therefore does not appear in the finite-dimensional factor space. If the Kac–Dynkin labels have been chosen appropriately (Kac 1978) so that Λ is the highest weight of a finite-dimensional factor space (so that supplementary conditions may apply), then $(\chi_j', \chi_j') = 0$. To determine the irreducible representations for these 'special' cases, we have found it necessary to examine explicitly matrix elements $\Pi(f_j^\mp, g_i^\pm)\chi_k$, where $\Pi(f_j^\mp, g_i^\mp)$ is some product of f_j^+, f_j^-, g_i^+ and g_i^-.

4.2. $C(2) \equiv OSp(2/2) \simeq A(1, 0)$

Dynkin diagram:

,

Cartan matrix:

$$(a_{ij}) = \begin{bmatrix} 0 & +1 \\ -1 & +2 \end{bmatrix}.$$

The odd generators are here designated as $\beta^{1\pm}$, $\beta_2^{1\pm}$. The even generators corresponding to the even positive and negative simple roots are $\alpha_2^\pm$. The generators of the Cartan subalgebra are h_1 and h_2. The Cartan generator corresponding to the O(2) generator is given by $k = 2h_1 - h_2$.

We designate the highest weight vector of an OSp(2/2) representation as Λ, with weight components $(a_1, a_2; b = 2a_1 - a_2)$ where $h_i\Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$, and $k\Lambda = \lambda(k)\Lambda =$

$b\Lambda$. Any OSp(2/2) representation can be uniquely decomposed in terms of O(2)×Sp(2) irreducible representations. In general we have four of these (see § 4.1). The weight components of the O(2)×Sp(2) highest weight vectors are given below

$$\begin{aligned} &\psi_1=\Lambda: && (a_1, a_2; b) \\ &\psi_2=\beta^{1-}\Lambda: && (a_1, a_2+1; b-1) \\ &\psi_3=\beta_2^{1-}\Lambda: && (a_1-1, a_2-1; b-1) \\ &\psi_4=\beta^{1-}\beta_2^{1-}\Lambda: && (a_1-1, a_2; b-2). \end{aligned} \tag{4.2}$$

Applying the procedure discussed in § 4.1, we find the corresponding O(2)×Sp(2) highest weight vectors are given by the following

$$\chi_1=\psi_1 \qquad \chi_2=\psi_2 \qquad \chi_3=\psi_3+\frac{1}{a_2+1}\alpha_2^-\chi_2 \qquad \chi_4=\psi_4. \tag{4.3}$$

As discussed in § 4.1, to find the conditions under which a state χ_i decouples from the highest weight we look for those conditions under which $(\chi_i, \chi_i)=0$. The inner products of the above states are given by the following:

$$\begin{aligned} &(\chi_1,\chi_1)_{A1,2}=(\chi_1,\chi_1)_{S1,2}=1 \\ &(\chi_2,\chi_2)_{A1}=-(\chi_2,\chi_2)_{A2}=-(\chi_2,\chi_2)_{S1}=(\chi_2,\chi_2)_{S2}=+a_1 \\ &(\chi_3,\chi_3)_{A1}=-(\chi_3,\chi_3)_{A2}=-(\chi_3,\chi_3)_{S1}=(\chi_3,\chi_3)_{S2}=-a_2(a_2-a_1+1)/(a_2+1) \\ &(\chi_4,\chi_4)_{A1,2}=-(\chi_4,\chi_4)_{S1}=-(\chi_4,\chi_4)_{S2}=-a_1(a_2-a_1+1). \end{aligned} \tag{4.4}$$

It can be seen that under the conditions (i) $a_1=0$ and (ii) $a_2-a_1+1=0$ the OSp(2/2) representation specified by the highest weight vector, Λ, is not irreducible and can be decomposed as shown in table 3. We require a_2 to be a non-negative integer for the representation to be finite dimensional.

From (4.4) we see that the only finite-dimensional irreducible representations defined on a graded Hilbert space are the following:

Star representations: (A1): $\{\chi_1,\chi_2,\chi_3,\chi_4\}$ if $b>a_2+2$, $\{\chi_1,\chi_2\}$ if $b=a_2+2$, $\{\chi_1\}$ if $a_2=b=0$; (A2): $\{\chi_1,\chi_2,\chi_3,\chi_4\}$ if $b+a_2<0$, $\{\chi_1,\chi_3\}$ if $a_2+b=0$.

Grade star representations: (S1): $\{\chi_1,\chi_3\}$ if $a_2+b=0$: (S2): $\{\chi_1,\chi_2,\chi_4\}$ if $a_2=0$ and $0<b<2$. $\{\chi_1,\chi_2\}$ if $\frac{1}{2}a_2-\frac{1}{2}b+1=0$, $\{\chi_1\}$ if $a_2=b=0$.

Table 3. Finite-dimensional atypical irreducible representations.

	Atypicality condition	Factor space	Invariant subspace
OSp(2/2):	$a_1=0$	χ_1, χ_3	χ_2, χ_4
	$a_2-a_1+1=0$	χ_1, χ_2	χ_3, χ_4
OSp(3/2):	$a_1=0$	χ_1	
	$a_2-a_1+1=0$	$\chi_1, \chi_2, \chi_3, \chi_5$	$\chi_4, \chi_6, \chi_7, \chi_8$
OSp(4/2):	$a_1=0$	χ_1	
	$a_2-a_1+1=0$	$\chi_1, \chi_2, \chi_4, \chi_5, \chi_7, \chi_8, \chi_{11}, \chi_{14}$	$\chi_3, \chi_6, \chi_9, \chi_{10}, \chi_{12}, \chi_{13}, \chi_{15}, \chi_{16}$
	$a_3-a_1+1=0$	$\chi_1, \chi_2, \chi_3, \chi_5, \chi_6, \chi_8, \chi_{10}, \chi_{13}$	$\chi_4, \chi_7, \chi_9, \chi_{11}, \chi_{12}, \chi_{14}, \chi_{15}, \chi_{16}$
	$a_2+a_3-a_1+2=0$	$\chi_1, \chi_2, \chi_3, \chi_4, \chi_6, \chi_7, \chi_8, \chi_{12}$	$\chi_5, \chi_9, \chi_{10}, \chi_{11}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}$

These results are in agreement with those of Scheunert *et al* (1977b) where the representation labels (b, q) correspond to $(\frac{1}{2}b-\frac{1}{2}, \frac{1}{2}a_2+\frac{1}{2})$ in the present notation.

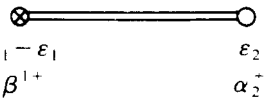
Taking $C(2)$ simply as the $n=2$ case of the general treatment of $C(n)$ as given in §§ 2 and 3 and Kac (1978) corresponds to taking the Cartan matrix as

$$(a_{ij}) = \begin{bmatrix} 0 & +2 \\ -1 & +2 \end{bmatrix},$$

With this, the value of the a_1 label in §§ 2 and 3 and Kac (1978) will be twice the value of the a_1 label in this section.

4.3. $B(1, 1) \equiv OSp(3/2)$

Dynkin diagram:



Cartan matrix:

$$(a_{ij}) = \begin{bmatrix} 0 & +1 \\ -2 & 2 \end{bmatrix}.$$

The odd generators are here designated as $\beta^{1\pm}$, $\beta_2^{1\pm}$, $\tilde{\beta}_2^{1\pm}$. The even generators are $\alpha_2^{\pm}$ corresponding to the even positive and negative simple roots. The 'hidden' Sp(2) generators are given by $\{\beta_2^{1\pm}, \beta_2^{1\pm}\}$. The Cartan generators are h_1 and h_2. The Cartan generator corresponding to the Sp(2) sector is given by $k=h_1-\frac{1}{2}h_2$.

We designate the highest weight vector of an OSp(3/2) representation as Λ, with weight components $(a_1, a_2; b=a_1-\frac{1}{2}a_2)$, where $h_i\Lambda=\lambda(h_i)\Lambda\equiv a_i\Lambda$, $k\Lambda=\lambda(k)\Lambda\equiv b\Lambda$. Any OSp(3/2) representation can be uniquely decomposed in terms of O(3)×Sp(2) irreducible representations. In general there will be eight of these (see § 4.1). The weight components of the O(3)×Sp(2) highest weight vectors are given below

$$\begin{aligned}
&\psi_1=\Lambda: && (a_1, a_2; b)\\
&\psi_2=\beta^{1-}\Lambda: && (a_1, a_2+2; b-1)\\
&\psi_3=\beta_2^{1-}\Lambda: && (a_1-1, a_2; b-1)\\
&\psi_4=\tilde{\beta}_2^{1-}\Lambda: && (a_1-2, a_2-2; b-1)\\
&\psi_5=\beta^{1-}\beta_2^{1-}\Lambda: && (a_1-1, a_2+2; b-2)\\
&\psi_6=\beta^{1-}\tilde{\beta}_2^{1-}\Lambda: && (a_1-2, a_2; b-2)\\
&\psi_7=\beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda: && (a_1-3, a_2-2; b-2)\\
&\psi_8=\beta^{1-}\beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda: && (a_1-3, a_2; b-3).
\end{aligned} \tag{4.5}$$

Applying the procedure discussed in § 4.1, we find the corresponding O(3)×Sp(2) highest weight vectors are given by the following:

$$\chi_1=\psi_1; \qquad \chi_2=\psi_2$$

$$\chi_3=\psi_3+\frac{2}{(a_2+2)}\alpha_2^-\chi_2$$

$$\begin{aligned}
\chi_4 &= \psi_4 + \frac{2}{a_2}\alpha_2^-\chi_3 - \frac{2}{(a_2+1)(a_2+2)}\alpha_2^-\alpha_2^-\chi_2 \\
\chi_5 &= \psi_5 \\
\chi_6 &= \psi_6 + \frac{2}{(a_2+2)}\alpha_2^-\chi_5 + \frac{(a_2-a_1)}{(a_2-2a_1)}\{\beta_2^{1-}, \beta_2^{1-}\}\chi_1 \\
\chi_7 &= \psi_7 + \frac{2}{a_2}\alpha_2^-\chi_6 - \frac{2}{(a_2+1)(a_2+2)}\alpha_2^-\alpha_2^-\chi_5 - \frac{1}{(a_2-2a_1)}\alpha_2^-\{\beta_2^{1-}, \beta_2^{1-}\}\chi_1 \\
\chi_8 &= \psi_8 - \frac{(a_2-a_1+2)}{(a_2-2a_1+2)}\{\beta_2^{1-}, \beta_2^{1-}\}\chi_3 + \frac{1}{(a_2+2)}\alpha_2^-\{\beta_2^{1-}, \beta_2^{1-}\}\chi_2.
\end{aligned} \tag{4.6}$$

As discussed in § 4.1 the conditions for which $(\chi_i, \chi_i) = 0$ are the conditions for which χ_i decouples from the highest weight. The inner products of the above states are given below

$$\begin{aligned}
&(\chi_1, \chi_1)_{S1,2} = 1 \\
&(\chi_2, \chi_2)_{S1} = -(\chi_2, \chi_2)_{S2} = -a_1 \\
&(\chi_3, \chi_3)_{S1} = -(\chi_3, \chi_3)_{S2} = +a_2(a_2-2a_1+2)/(a_2+2) \\
&(\chi_4, \chi_4)_{S1} = -(\chi_4, \chi_4)_{S2} = +4(a_2-1)(a_2-a_1+1)/(a_2+1), \qquad a_2 \neq 0 \\
&(\chi_5, \chi_5)_{S1} = (\chi_5, \chi_5)_{S2} = +a_1(a_2-2a_1+2) \\
&(\chi_6, \chi_6)_{S1} = (\chi_6, \chi_6)_{S2} = +4a_1a_2(a_2-a_1+1)(a_2-2a_1+2)/[(a_2+2)(a_2-2a_1)], \qquad b \neq 0 \\
&(\chi_7, \chi_7)_{S1} = (\chi_7, \chi_7)_{S2} = -4(a_2-1)(a_2-a_1+1)(a_2-2a_1+2)/(a_2+1), \qquad a_2 \neq 0, b \neq 0 \\
&(\chi_8, \chi_8)_{S1} = -(\chi_8, \chi_8)_{S2} = 4a_1(a_2-a_1+1)(a_2-2a_1+4), \qquad b \neq 1.
\end{aligned} \tag{4.7}$$

It can be seen that under the condition $a_2 - a_1 + 1 = 0$, the OSp(3/2) representation specified by the highest weight vector, Λ, is not irreducible and can be decomposed as shown in table 3. As discussed in § 4.1, if $b = 0, 1$ or $a_2 = 0$, then (4.6) must be modified as per (4.1a). If $b = 0$, then the representation is atypical and to obtain a finite-dimensional representation we must also impose the supplementary condition $a_2 = 0$ (Kac 1978). This gives the singlet, χ_1, as the only irreducible finite-dimensional representation. For the 'special' cases $a_2 = 0$ or $b = 1$, we find the only finite-dimensional irreducible representations occur as factor spaces. These are $a_2 = 0$, $\{\chi_1, \chi_2, \chi_5, \chi_8\}$, the adjoint is obtained from this by setting $b = 2$; $b = 1$, $\{\chi_1, \chi_2, \chi_4\}$. If $a_2 = 0$ and $b = 1$, we obtain the fundamental $\{\chi_2, \chi_2\}$. The decompositions for all atypical, irreducible, finite-dimensional representations are given in table 3. For the existence of a finite-dimensional representation, we require a_2 and b to be non-negative integers.

From (4.7) and the above discussion we see that the only finite-dimensional irreducible representations defined on a graded Hilbert space are the following grade star representations:

(S1) $\{\chi_1\}$ if $a_2 = b = 0$

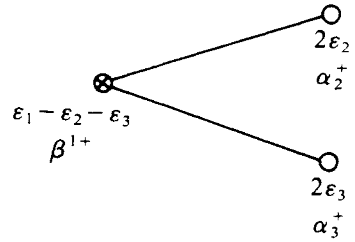
(S2) $\{\chi_1\}$ if $a_2 = b = 0$;

$\{\chi_1, \chi_2\}$ if $b = 1$; $a_2 = 0, 1$.

We note that all the above finite-dimensional irreducible representations were found by the superfield approach (Farmer and Jarvis 1983).

4.4. $D(2, 1) \equiv OSp(4/2)$:

Dynkin diagram:



Cartan matrix:

$$(a_{ij}) = \begin{bmatrix} 0 & +1 & +1 \\ -1 & +2 & 0 \\ -1 & 0 & +2 \end{bmatrix}.$$

The odd generators are designated here as

$$\beta^{1\pm}, \quad \beta_2^{1\pm}, \quad \beta_3^{1\pm}, \quad \tilde{\beta}_2^{1\pm}.$$

The even generators corresponding to the even positive and negative simple roots are $\alpha_2^{\pm}$ and $\alpha_3^{\pm}$. The 'hidden' Sp(2) generators are given by $\{\beta_2^{1\pm}, \beta_3^{1\pm}\}$. The generators of the Cartan subalgebra are h_1, h_2 and h_3. The Cartan generator corresponding to the Sp(2) sector is given by $k = \frac{1}{2}(2h_1 - h_2 - h_3)$.

We designate the highest weight vector of an OSp(4/2) representation as Λ, with weight components $(a_1, a_2, a_3; b = \frac{1}{2}(2a_1 - a_2 - a_3))$, where $h_i\Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$ and $k\Lambda = \lambda(k)\Lambda \equiv b\Lambda$. Any OSp(4/2) representation can be uniquely decomposed in terms of $O(4) \times Sp(2)$ irreducible representations. In general we have 16 of these (see § 4.1). The weight components of the $O(4) \times Sp(2)$ highest weight vectors are given below

$$\begin{array}{ll}
\psi_1 = \Lambda: & (a_1, a_2, a_3; b) \\
\psi_2 = \beta^{1-}\Lambda: & (a_1, a_2+1, a_3+1; b-1) \\
\psi_3 = \beta_2^{1-}\Lambda: & (a_1-1, a_2-1, a_3+1; b-1) \\
\psi_4 = \beta_3^{1-}\Lambda: & (a_1-1, a_2+1, a_3-1; b-1) \\
\psi_5 = \tilde{\beta}_2^{1-}\Lambda: & (a_1-2, a_2-1, a_3-1; b-1) \\
\psi_6 = \beta^{1-}\beta_2^{1-}\Lambda: & (a_1-1, a_2, a_3+2; b-2) \\
\psi_7 = \beta^{1-}\beta_3^{1-}\Lambda: & (a_1-1, a_2+2, a_3; b-2) \\
\psi_8 = \beta^{1-}\tilde{\beta}_2^{1-}\Lambda: & (a_1-2, a_2, a_3; b-2) \\
\psi_9 = \beta_2^{1-}\beta_3^{1-}\Lambda: & (a_1-2, a_2, a_3; b-2) \\
\psi_{10} = \beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda: & (a_1-3, a_2-2, a_3; b-2) \\
\psi_{11} = \beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda: & (a_1-3, a_2, a_3-2; b-2) \\
\psi_{12} = \beta^{1-}\beta_2^{1-}\beta_3^{1-}\Lambda: & (a_1-2, a_2+1, a_3+1; b-3) \\
\psi_{13} = \beta^{1-}\beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda: & (a_1-3, a_2-1, a_3+1; b-3)
\end{array} \tag{4.8}$$

$$\psi_{14}=\beta^{1-}\beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda: \qquad (a_1-3, a_2+1, a_3-1; b-3)$$

$$\psi_{15}=\beta_2^{1-}\beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda: \qquad (a_1-4, a_2-1, a_3-1; b-3)$$

$$\psi_{16}=\beta^{1-}\beta_2^{1-}\beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda: \quad (a_1-4, a_2, a_3; b-4).$$

Applying the procedure discussed in § 4.1, we find the corresponding O(4) × Sp(2) highest weight vectors are given by the following:

$$
\begin{aligned}
\chi_1 &= \psi_1\\
\chi_2 &= \psi_2\\
\chi_3 &= \psi_3+\frac{1}{(a_2+1)}\alpha_2^-\chi_2\\
\chi_4 &= \psi_4+\frac{1}{(a_3+1)}\alpha_3^-\chi_2\\
\chi_5 &= \psi_5+\frac{1}{(a_2+1)}\alpha_2^-\chi_4+\frac{1}{(a_3+1)}\alpha_3^-\chi_3-\frac{1}{(a_2+1)(a_3+1)}\alpha_2^-\alpha_3^-\chi_2\\
\chi_6 &= \psi_6\\
\chi_7 &= \psi_7\\
\tilde{\chi}_8 &= \psi_8+\frac{1}{(a_2+2)}\alpha_2^-\chi_7+\frac{1}{(a_3+2)}\alpha_3^-\chi_6+\frac{(a_2+a_3-a_1)}{(a_2+a_3-2a_1)}\{\beta_2^{1-},\beta_3^{1-}\}\chi_1\\
\tilde{\chi}_9 &= \psi_9+\frac{1}{(a_2+2)}\alpha_2^-\chi_7-\frac{1}{(a_3+2)}\alpha_3^-\chi_6-\frac{(a_3-a_1)}{(a_2+a_3-2a_1)}\{\beta_2^{1-},\beta_3^{1-}\}\chi_1\\
\chi_{10} &= \psi_{10}+\frac{1}{a_2}\alpha_2^-\tilde{\chi}_9+\frac{1}{a_2}\alpha_2^-\tilde{\chi}_8-\frac{1}{(a_2+1)(a_2+2)}\alpha_2^-\alpha_2^-\chi_7\\
&\quad -\frac{1}{(a_2+a_3-2a_1)}\alpha_2^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_1\\
\chi_{11} &= \psi_{11}-\frac{1}{a_3}\alpha_3^-\tilde{\chi}_9+\frac{1}{a_3}\alpha_3^-\tilde{\chi}_8-\frac{1}{(a_3+1)(a_3+2)}\alpha_3^-\alpha_3^-\chi_6\\
&\quad -\frac{1}{(a_2+a_3-2a_1)}\alpha_3^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_1 \qquad (4.9)\\
\chi_{12} &= \psi_{12}-\frac{(a_3-a_1+1)}{(a_2+a_3-2a_1+2)}\{\beta_2^{1-},\beta_3^{1-}\}\chi_2\\
\chi_{13} &= \psi_{13}+\frac{1}{(a_2+1)}\alpha_2^-\chi_{12}-\frac{(a_2+a_3-a_1+2)}{(a_2+a_3-2a_1+2)}\{\beta_2^{1-},\beta_3^{1-}\}\chi_3\\
&\quad +\frac{(a_3-a_1+1)}{(a_2+1)(a_2+a_3-2a_1+2)}\alpha_2^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_2\\
\chi_{14} &= \psi_{14}-\frac{1}{(a_3+1)}\alpha_3^-\chi_{12}-\frac{(a_2+a_3-a_1+2)}{(a_2+a_3-2a_1+2)}\{\beta_2^{1-},\beta_3^{1-}\}\chi_4\\
&\quad +\frac{(a_2-a_1+1)}{(a_3+1)(a_2+a_3-2a_1+2)}\alpha_3^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_2
\end{aligned}
$$

$$\chi_{15}=\psi_{15}+\frac{1}{(a_2+1)}\alpha_2^-\chi_{14}+\frac{1}{(a_3+1)}\alpha_3^-\chi_{13}+\frac{1}{(a_2+1)(a_3+1)}\alpha_2^-\alpha_3^-\chi_{12}$$
$$-\frac{(a_3-a_1+1)}{(a_2+a_3-2a_1+2)}\{\beta_2^{1-},\beta_3^{1-}\}\chi_5$$
$$+\frac{(a_2+a_3-a_1+2)}{(a_2+1)(a_2+a_3-2a_1+2)}\alpha_2^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_4$$
$$-\frac{a_1}{(a_3+1)(a_2+a_3-2a_1+2)}\alpha_3^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_3$$
$$-\frac{(a_2-a_1+1)}{(a_2+1)(a_3+1)(a_2+a_3-2a_1+2)}\alpha_2^-\alpha_3^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_2$$
$$\chi_{16}=\psi_{16}+\frac{(a_2+a_3-a_1+4)}{(a_2+a_3-2a_1+4)}\{\beta_2^{1-},\beta_3^{1-}\}\tilde{\chi}_9-\frac{(a_3-a_1+2)}{(a_2+a_3-2a_1+4)}\{\beta_2^{1-},\beta_3^{1-}\}\tilde{\chi}_8$$
$$+\frac{1}{(a_3+2)}\alpha_3^-\{\beta_2^{1-},\beta_3^{1-}\}\chi_6$$
$$+\frac{(a_1^2+a_3^2-a_1a_2-2a_1a_3+a_2a_3-a_1+2a_3)}{(a_2+a_3-2a_1)(a_2+a_3-2a_1+2)}\{\beta_2^{1-},\beta_3^{1-}\}\{\beta_2^{1-},\beta_3^{1-}\}\chi_1.$$

Examination of the weights of the above states reveals a degeneracy in the sense that ψ_8 and ψ_9 possess the same weight. Since the orthogonalisation procedure we have used does not allow us to overcome this multiplicity problem, we have been obliged to determine the irreducible spaces to which the corresponding $O(4)\times Sp(2)$ highest weight vectors, χ_8 and χ_9, belong by mapping from states in the invariant subspace to linear combinations of $\tilde{\chi}_8$ and $\tilde{\chi}_9$. We can then determine from the nature of these linear combinations whether both, none, or only one of χ_8 and χ_9 belong to the invariant subspace. The inner products, (χ_i, χ_i), of the remaining states are given below:

$$\begin{aligned}
&(\chi_1,\chi_1)_{S1,2}=1\\
&(\chi_2,\chi_2)_{S1}=-(\chi_2,\chi_2)_{S2}=-a_1\\
&(\chi_3,\chi_3)_{S1}=-(\chi_3,\chi_3)_{S2}=+a_2(a_2-a_1+1)/(a_2+1)\\
&(\chi_4,\chi_4)_{S1}=-(\chi_4,\chi_4)_{S2}=+a_3(a_3-a_1+1)/(a_3+1)\\
&(\chi_5,\chi_5)_{S1}=-(\chi_5,\chi_5)_{S2}=+a_2a_3(a_2+a_3-a_1+2)/[(a_2+1)(a_3+1)]\\
&(\chi_6,\chi_6)_{S1}=(\chi_6,\chi_6)_{S2}=+a_1(a_2-a_1+1)\\
&(\chi_7,\chi_7)_{S1}=(\chi_7,\chi_7)_{S2}=+a_1(a_3-a_1+1)\\
&(\chi_{10},\chi_{10})_{S1}=(\chi_{10},\chi_{10})_{S2}=-(a_2-a_1+1)(a_2+a_3-a_1+2)(a_2-1)/(a_2+1),\\
&\qquad a_2\neq 0,\ b\neq 0\\
&(\chi_{11},\chi_{11})_{S1}=(\chi_{11},\chi_{11})_{S2}=-(a_3-a_1+1)(a_2+a_3-a_1+2)(a_3-1)/(a_3+1),\\
&\qquad a_3\neq 0,\ b\neq 0\\
&(\chi_{12},\chi_{12})_{S1}=-(\chi_{12},\chi_{12})_{S2}=a_1(a_2-a_1+1)(a_3-a_1+1)\\
&\qquad\times(a_2+a_3-2a_1+4)/(a_2+a_3-2a_1+2),\qquad b\neq 1
\end{aligned}\tag{4.10}$$

$$(\chi_{13},\chi_{13})_{S1}=-(\chi_{13},\chi_{13})_{S2}=a_1a_2(a_2-a_1+1)(a_2+a_3-2a_1+4)\times(a_2+a_3-a_1+2)/[(a_2+1)(a_2+a_3-2a_1+2)],\qquad b\neq 1$$

$$(\chi_{14},\chi_{14})_{S1}=-(\chi_{14},\chi_{14})_{S2}=a_1a_3(a_3-a_1+1)(a_2+a_3-2a_1+4)\times(a_2+a_3-a_1+2)/[(a_3+1)(a_2+a_3-2a_1+2)],\qquad b\neq 1$$

$$(\chi_{15},\chi_{15})_{S1}=-(\chi_{15},\chi_{15})_{S2}=-a_2a_3(a_2-a_1+1)(a_3-a_1+1)(a_2+a_3-a_1+2)\times(a_2+a_3-2a_1+4)/[(a_2+1)(a_3+1)(a_2+a_3-2a_1+2)],\qquad b\neq 1$$

$$(\chi_{16},\chi_{16})_{S1}=(\chi_{16},\chi_{16})_{S2}=a_1(a_2-a_1+1)(a_3-a_1+1)(a_2+a_3-a_1+2)\times(a_2+a_3-2a_1+6)/(a_2+a_3-2a_1+2),\qquad b\neq 0,2.$$

It can be seen that under the conditions (i) $a_2-a_1+1=0$, (ii) $a_3-a_1+1=0$, and (iii) $a_2+a_3-a_1+2=0$, the OSp(4/2) representation specified by the highest weight vector, Λ, is not irreducible and can be decomposed as shown in table 3. As discussed in § 4.1, if $b=0,1,2$ or $a_2=0$, or $a_3=0$, then (4.9) must be modified as per (4.1*a*). If $b=0$, then to obtain a finite-dimensional representation we must also impose the supplementary conditions $a_2=a_3=0$ (Kac 1978). This gives the singlet, χ_1, as the only irreducible, finite-dimensional representation. Similarly, if $b=1$, then we must impose the supplementary condition $a_2=a_3$. This gives $\{\chi_1,\chi_2,\chi_5\}$ as the only finite-dimensional, irreducible representation. Other 'special' cases we note are: if $b=2$, or $a_2=0$, or $a_3=0$, then one of χ_8 or χ_9 is part of the infinite-dimensional subspace: if $a_2=a_3=0$, then both χ_8 and χ_9 belong to the infinite-dimensional subspace. For the following cases table 3 must be modified as specified: if condition (iii) above is imposed and either $a_2=0$ or $a_3=0$, or both, then χ_9 is part of the infinite-dimensional subspace: if condition (ii) is imposed and $a_2=0$, or if condition (i) is imposed and $a_3=0$, then χ_9 is part of the infinite-dimensional subspace. If $b=1$ and $a_2=a_3=0$, we obtain the fundamental $\{\chi_1,\chi_2\}$. If $b=2$ and $a_2=a_3=0$, we obtain the adjoint $\{\chi_1,\chi_2,\chi_6,\chi_7\}$. The decompositions for all atypical, finite-dimensional, irreducible representations are given in table 3. For the existence of a finite-dimensional representation we require a_2, a_3 and b to be non-negative integers.

From (4.10) and the above discussion, we see that the only finite-dimensional irreducible representations defined on a graded Hilbert space are the following grade star representations:

(S1): $\{\chi_1\}$ if $a_2=a_3=b=0$;

(S2): $\{\chi_1,\chi_2,\chi_5\}$ if $b=1$,
$\{\chi_1,\chi_2,\chi_6,\chi_7\}$ if $b=2$ and $a_2=a_3=0$,
$\{\chi_1\}$ if $a_2=a_3=b=0$.

We note that the finite-dimensional, irreducible representations found by the superfield approach (Farmer and Jarvis 1983) are all confirmed in the above analysis.

Appendix 1

In the following we present a diagrammatic proof that the choice (3.5), $\tilde{\mu}$, for ξ and $\beta=(0)$ in (3.1*a*) uniquely determines the highest weight vector, Λ, for $B(m,n)$ and $D(m,n)$. Given the selection criteria, for the diagram corresponding to Λ, which are presented in § 3.1, this proof amounts to showing that if $\xi = (\widetilde{\mu + \chi})$ has $(n+x)$ columns where χ corresponds to the final x rows in $\tilde{\xi}$, then all the partitions in the series $\langle \tilde{\xi}/B \rangle$ modify in Sp$(2n)$ to a partition of rank $< |\mu|$. The rank of a partition (ρ) we designate as $|\rho|$.

We first note that if (χ) is a partition of the form

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ a_1+1 & a_2+1 & \ldots & a_r+1 \end{pmatrix}$$

in Frobenius notation (King 1975), then $\langle \mu + \chi \rangle$ modifies to $(-1)^{|\chi|/2} \langle \mu \rangle$; otherwise it modifies to partitions of rank $< |\mu|$. Our proof is by induction in which we show that if $\langle (\mu + \chi)/B \rangle$ contains no diagram of rank $|\mu|$ then $\langle [\mu + (\chi + 2)]/B \rangle$ contains no diagram of rank $|\mu|$, where $(\chi + 2)$ is any partition for which $|(\chi+2)| = |\chi| + 2$ and $\mu + (\chi + 2)$ is a proper diagram. Since, in Sp$(2n)$, modification involves removing a hook of length $h = 2(P - n - 1) \geq 0$ (King 1975), then unless $|\chi|$ is even $\langle (\mu + \chi)/B \rangle$ will modify to partitions of rank $< |\mu|$. Consider now

$$\left\{ \frac{\mu + (\chi+2)}{A} \right\} = \{\mu + (\chi+2)\} + \sum \{\mu + (\chi+2)''\} + \sum \{\mu'' + (\chi+2)\} + \sum \{\mu' + (\chi+2)'\} \tag{A1.1}$$

where $|\mu'| \leq |\mu| - 1$, $|\mu''| \leq |\mu| - 2$, $|(\chi+2)'| \leq |\chi| + 1$, $0 \leq |(\chi+2)''| \leq |\chi|$, and A is the s-function series $a = \sum_\alpha (-1)^{|\alpha|/2}\{\alpha\}$. We now divide both sides by B and use $AB = 1 = \{0\}$ to give

$$\left\{ \frac{\mu + (\chi+2)}{B} \right\} = \{\mu + (\chi+2)\} - \sum \left\{ \frac{\mu + (\chi+2)''}{B} \right\} - \sum \left\{ \frac{\mu'' + (\chi+2)}{B} \right\} - \sum \left\{ \frac{\mu' + (\chi+2)'}{B} \right\}. \tag{A1.2}$$

Examining (A1.2) we see that the final two terms explicitly modify to partitions of rank $< |\mu|$. Considering the first and second terms, we note two possibilities:

(i) $(\chi + 2)$ is not of form (α). In this case, $2 \leq |(\chi+2)''| \leq |\chi|$ and by our assertion $\langle (\mu + (\chi+2)'')/B \rangle$ will modify only to partitions of rank $< |\mu|$. Also as noted earlier the first term will modify to a partition of rank $< |\mu|$.

(ii) $(\chi + 2)$ is of form (α). For this case in (A1.1) we have

$$\sum \{\mu + (\chi+2)''\} = (-1)^{|\alpha|/2}\{\mu\} + \sum \{\mu + (\chi+2)'''\}$$

where now $2 \leq |(\chi+2)'''| \leq |\chi|$. The second term in (A1.2) therefore takes the form

$$(-1)^{|\alpha|/2} \left\{ \frac{\mu}{B} \right\} + \sum \left\{ \frac{\mu + (\chi+2)'''}{B} \right\}.$$

By our assertion the last term here modifies only to partitions of rank $< |\mu|$ and the first term is explicitly of rank $< |\mu|$ except for $\beta = \{0\}$. The first term in (A1.2) modifies however to $(-1)^{|\alpha|/2}\langle \mu \rangle$. Thus the only terms contributing to $\langle \mu \rangle$ in (A1.2) will cancel.

To complete the proof we need only show that for $|\chi| = 2$, $\langle (\mu + \chi)/B \rangle$ modifies only to partitions of rank $< |\mu|$. There are only two possibilities:

(i)

$$\begin{aligned} \chi = (1^2): \quad \langle (\mu + (1^2))/B \rangle &= \langle \mu + (1^2) \rangle + \langle \mu \rangle + \sum \langle \mu' + (1) \rangle + \sum \langle \mu'' + (1^2) \rangle + \sum \langle \mu''' \rangle \\ &\to -\langle \mu \rangle + \langle \mu \rangle - \sum \langle \mu'' \rangle + \sum \langle \mu''' \rangle \\ &= \sum \langle \mu''' \rangle - \sum \langle \mu'' \rangle \qquad \text{where } |\mu''|, |\mu'''| < |\mu|. \end{aligned}$$

(ii)

$$\chi=(2): \quad \langle(\mu+(2))/B\rangle=\langle\mu+(2)\rangle+\sum\langle\mu'+(1)\rangle$$
$$+\sum\langle\mu''+(2)\rangle+\sum\langle\mu'''\rangle$$
$$\rightarrow\sum\langle\mu'''\rangle \qquad \text{where } |\mu'''|<|\mu|.$$

Appendix 2

The adjoints (†) and superadjoints (‡) of all even root vectors corresponding to simple roots and of all generators in the Cartan subalgebra are defined as follows:

$$(\alpha_j^{\pm})^{\dagger}=(\alpha_j^{\pm})^{\ddagger}=\alpha_j^{\mp}, \qquad (h_i)^{\dagger}=(h_i)^{\ddagger}=h_i.$$

The adjoints and superadjoints of the odd root vectors can each be defined in two ways which we designate as $A\sigma$ and $S\sigma$ respectively, where $\sigma=1$ or 2. If $\beta^{\pm}$ is the positive or negative odd root vector corresponding to the odd simple root as discussed in § 2, then we define

$$(\beta^{\pm})^{\dagger}=\pm(\beta^{\pm})^{\ddagger}=(-1)^{\sigma}\beta^{\mp}.$$

The adjoints and superadjoints of the remaining odd generators for $B(m, n)$, $D(m, n)$ and $C(n)$ can now be evaluated from their definitions in § 2. For example, for $C(n)$:

$$(\beta_j^{\pm})^{\dagger}=(-1)^{j+\sigma}\beta_j^{\mp} \qquad (\tilde{\beta}_{n-k}^{\pm})^{\dagger}=(-1)^{n+k+\sigma}\tilde{\beta}_{n-k}^{\mp}$$
$$(\beta_j^{\pm})^{\ddagger}=\pm(-1)^{j+\sigma+1}\beta_j^{\mp} \qquad (\tilde{\beta}_{n-k}^{\pm})^{\ddagger}=\pm(-1)^{n+k+\sigma}\tilde{\beta}_{n-k}^{\mp}$$

where $1\leqslant j\leqslant n$, $1\leqslant k\leqslant n-2$.

We note that for $B(m, n)$ and $D(m, n)$, but not for $C(n)$, the 'hidden' even Sp($2n$) generator $\{\beta_a^{\pm}, \beta_b^{\pm}\}$ defined in § 2 transforms as

$$\{\beta_a^{\pm}, \beta_b^{\pm}\}^{\ddagger}=-\{\beta_a^{\mp}, \beta_b^{\mp}\} \qquad \text{and} \qquad \{\beta_a^{\pm}, \beta_b^{\pm}\}^{\dagger}=+\{\beta_a^{\mp}, \beta_b^{\mp}\}$$

corresponding to compact and non-compact real forms of Sp($2n$) in the superadjoint and adjoint cases, respectively.

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